A Max-Algebra Approach to Modeling and Simulation of Tandem Queueing Systems*

N. K. Krivulin†‡

Abstract

Max-algebra models of tandem single-server queueing systems with both finite and infinite buffers are developed. The dynamics of each system is described by a linear vector state equation similar to those in the conventional linear systems theory, and it is determined by a transition matrix inherent in the system. The departure epochs of a customer from the queues are considered as state variables, whereas its service times are assumed to be system parameters. We show how transition matrices may be calculated from the service times, and present the matrices associated with particular models. We also give a representation of system performance measures including the system time and the waiting time of customers, associated with the models. As an application, both serial and parallel simulation procedures are presented, and their performance is outlined.

Key-Words: max-algebra, tandem queues, dynamic state equation, performance measure, parallel simulation algorithm.

1 Introduction

In the analysis of queueing systems, algebra models arise naturally from the recursive equations of the Lindley type, which present a formalism widely used for the representation of dynamics of a variety of queueing systems models. There are the recursive equations designed to describe the $G/G/m$ queue, closed and open tandem queueing systems which may have both infinite and finite buffers, and queueing networks with deterministic routing (see e.g., [1, 2, 3, 4, 5, 6]). These equations, which allow the dynamics of

*Mathematical and Computer Modelling, 1995. Vol. 22, no. 3, pp. 25-31.

†Faculty of Mathematics and Mechanics, St. Petersburg State University, 28 Universitetsky Ave., St. Petersburg, 198504, Russia, nkk@math.spbu.ru

‡The writing of this paper was completed while the author was visiting the Center for Economic Research (CentER) at Tilburg University, Tilburg, The Netherlands. The author thanks the CentER for its hospitality. He is also very much grateful to J. P. C. Kleijnen for valuable comments and suggestions which allow the author to improve the clarity of presentation.

a queueing system to be represented in a convenient and unified way well suited to analytical treatments, also provide the basis for the development of efficient procedures of queueing system simulation [1, 2, 3, 7].

Since recursive equations often involve only the operations of arithmetic addition and maximization, they offer the prospect of the representation of queueing system models in terms of the *max-algebra theory* [8, 9, 10]. The implementation of max-algebra allows one to rewrite the recursive equations as linear scalar and vector algebraic equations [11], which are actually almost identical to those in the conventional linear system theory. The benefits of the max-algebra approach in the analysis of queueing systems are twofold: first, it gives us the chance to exploit results of the conventional linear algebra, which have been reformulated, and are now available in the max-algebra. The classical results already reformulated and proved in the max-algebra include, in particular, the solution of the eigenvalue problem, the Cayley-Hamilton theorem, and Cramer's rule [8, 9, 10, 12].

Other benefits have a direct relationship to computational aspects of simulation. In fact, the algebraic models of queues lead to matrix-vector max-algebra multiplications as the basis of simulation procedures [2]. New possibilities then arise in queueing system simulation to employ efficient computational methods and algorithms available in numerical algebra, including those designed for implementation on parallel and vector processors.

In this paper we develop max-algebra models of open and closed tandem single-server queueing systems which may have both infinite and finite buffers, and we give related representations of system performance measures. We start with preliminary algebraic definitions in Section 2 which also includes a technical lemma underlying the development of models in later sections. In Section 3, the dynamics of open and closed systems with infinite buffers is described by a vector state equation which is determined by a transition matrix inherent in the system. The departure epochs of a customer from the queues are considered as state variables, whereas its service times are assumed to be system parameters. We show how transition matrices may be calculated from the service times, and present the matrices associated with certain particular models.

Section 4 extends the dynamic equation to cover open tandem systems with finite buffers, which operate under both manufacturing and communication blocking rules. The representations of system performance measures including the system times and the waiting times of customers, associated with the models are given in Section 5. In Section 6, we present serial and parallel simulation algorithms based on the algebraic models, and outline their performance. Finally, Section 7 gives conclusions.

2 Preliminary Algebraic Definitions and Results

In this section we briefly outline basic facts about matrix max-algebra, which underlie the algebraic models and methods of queueing system simulation, presented in the subsequent sections. Further details concerning the max-algebra and its applications can be found in survey papers [10, 12]. A thorough theoretical analysis of this algebra and related algebraic systems is given in [8, 9].

We start with the max-algebra of real numbers, which is the system $(\underline{\mathbb{R}}, \oplus, \otimes)$, where $\underline{\mathbb{R}} = \mathbb{R} \cup \{\varepsilon\}$ with $\varepsilon = -\infty$, and

$$x \oplus y = \max(x, y), \qquad x \otimes y = x + y$$

for any $x, y \in \mathbb{R}$.

It is easy to see that these new operations, namely addition $\oplus$ and product $\otimes$, possess the following properties:

Associativity:	$x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
	$x \otimes (y \otimes z) = (x \otimes y) \otimes z$;
Commutativity:	$x \oplus y = y \oplus x, \quad x \otimes y = y \otimes x$;
Distributivity:	$x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$;
Idempotency of Addition:	$x \oplus x = x$.

With $e = 0$, we further have

Null and Identity Elements:	$x \oplus \varepsilon = \varepsilon \oplus x = x, \quad x \otimes e = e \otimes x = x$;
Absorption Rule:	$x \otimes \varepsilon = \varepsilon \otimes x = \varepsilon$.

Clearly, in the max-algebra these properties allow ordinary algebraic manipulation of expressions involving the max-algebra operations to be performed under the usual conventions regarding brackets and precedence of $\otimes$ over $\oplus$.

Note finally that in the max-algebra, for each $x \in \mathbb{R}$, there exists its multiplicative inverse x^{-1} such that $x \otimes x^{-1} = x^{-1} \otimes x = e$. This is the usual arithmetic inverse which satisfies, in particular, the evident condition

$$(x \otimes y)^{-1} = x^{-1} \otimes y^{-1}$$

for all $x, y \in \mathbb{R}$.

2.1 Max-algebra of Matrices

The scalar max-algebra is extended to the max-algebra of matrices in the regular way. Specifically, for any square $(n \times n)$-matrices $A = (a_{ij})$ and

$B = (b_{ij})$ with entries in $\underline{\mathbb{R}}$, the elements of the matrices $C = A \oplus B$ and $D = A \otimes B$ are calculated as

$$c_{ij} = a_{ij} \oplus b_{ij}, \qquad \text{and} \qquad d_{ij} = \sum_{k=1}^{n}{}_{\oplus}\, a_{ik} \otimes b_{kj},$$

where $\sum_{\oplus}$ denotes the iterated operation $\oplus$, $i = 1, \ldots, n$; $j = 1, \ldots, n$. Similarly, the multiplication of a matrix by a scalar, as well as the operations of both matrix-vector multiplication and vector addition may be routinely defined.

As in the scalar max-algebra, there are null and unit elements in the matrix algebra, defined respectively as

$$\mathcal{E} = \begin{pmatrix} \varepsilon & \ldots & \varepsilon \\ \vdots & \ddots & \vdots \\ \varepsilon & \ldots & \varepsilon \end{pmatrix}, \qquad E = \begin{pmatrix} e & & \varepsilon \\ & \ddots & \\ \varepsilon & & e \end{pmatrix}.$$

One can easily see that for any square matrix A, it holds

$$\mathcal{E} \otimes A = A \otimes \mathcal{E} = \mathcal{E}, \qquad E \otimes A = A \otimes E = A, \qquad \mathcal{E} \oplus A = A \oplus \mathcal{E} = A.$$

It is not difficult to verify that the other properties of scalar operations $\oplus$ and $\otimes$, with the exception of the commutativity of multiplication, are also extended to the matrix algebra. Similar to the conventional matrix algebra, matrix multiplication in the max-algebra is not commutative in general. Furthermore, the multiplicative inverse does not generally exists in this matrix algebra. However, one can easily obtain the inverse of any diagonal square matrix A with entries not equal to ε only on the diagonal. It is clear that with

$$A = \begin{pmatrix} a_1 & & \varepsilon \\ & \ddots & \\ \varepsilon & & a_n \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} a_1^{-1} & & \varepsilon \\ & \ddots & \\ \varepsilon & & a_n^{-1} \end{pmatrix},$$

where $a_i > \varepsilon$ for all $i = 1, \ldots, n$, we have $A \otimes A^{-1} = A^{-1} \otimes A = E$.

Finally, the properties of matrix multiplication allow us to exploit the symbol A^p with a square matrix A and nonnegative integer p, as used in the conventional algebra:

$$A^e = E, \qquad A^p = A^{p-1} \otimes A = \underbrace{A \otimes \cdots \otimes A}_{p \text{ times}} \quad \text{for } p \geq 1.$$

2.2 A Linear Algebraic Equation

Let us now examine a vector equation which will be encountered below in algebraic representations of tandem queueing system dynamics. For given

$(n \times n)$-matrix A and column n-vector $\boldsymbol{b}$, we consider the implicit equation in the n-vector $\boldsymbol{x}$

$$\boldsymbol{x} = A \otimes \boldsymbol{x} \oplus \boldsymbol{b}, \tag{1}$$

which is generally identified analogous to the conventional linear algebra as a linear equation.

The next lemma offers the solution of (1) for a particular class of matrices A. A detailed investigation of this and other linear equations in the general case can be found in [9].

Lemma 1. *If there exists a nonnegative integer p such that $A^p = \mathcal{E}$, then (1) has the unique solution*

$$\boldsymbol{x} = \sum_{i=0}^{p-1}{}^{\oplus} A^i \otimes \boldsymbol{b}. \tag{2}$$

Proof. Recurrent substitutions of $\boldsymbol{x}$ from (1) into the right-hand side of (1) and trivial algebraic manipulations give

$$\begin{aligned} \boldsymbol{x} = A \otimes \boldsymbol{x} \oplus \boldsymbol{b} = A \otimes (A \otimes \boldsymbol{x} \oplus \boldsymbol{b}) \oplus \boldsymbol{b} = A^2 \otimes \boldsymbol{x} \oplus (E \oplus A) \otimes \boldsymbol{b} \\ = \cdots = A^p \otimes \boldsymbol{x} \oplus (E \oplus A \oplus \cdots \oplus A^{p-1}) \otimes \boldsymbol{b}. \end{aligned}$$

With the condition $A^p = \mathcal{E}$, we immediately arrive at (2). It is also evident from the above calculations that the obtained solution is unique. □

As examples of the matrix A satisfying the condition of Lemma 1, we consider either lower or upper triangular matrices which have, in addition, the entries equal to ε on the main diagonal. Specifically, it is not difficult to verify that for the matrix A with entries

$$a_{ij} = \begin{cases} \alpha_i > \varepsilon, & \text{if } i = j+1 \\ \varepsilon, & \text{otherwise,} \end{cases}$$

which will appear in the next sections, it holds that $A^p \neq \mathcal{E}$, for $p = 1, \ldots, n-1$, and $A^n = \mathcal{E}$.

3 Representation of Tandem System Dynamics

We start with the scalar max-algebra equation representing the dynamics of a single-server queue, and then extend it to vector equations associated with tandem systems of queues. The queue is assumed to have a buffer with infinite capacity, and to operate under the first-come, first-served queue discipline.

We denote the k^{th} arrival epoch to the queue and the k^{th} departure epoch from the queue by $a(k)$ and $d(k)$ respectively. The service time

of customer k is represented by τ_k. Under the conditions that the queue starts operating at time zero and it has no customers at the initial time, the dynamics of the system can be readily described in terms of $a(k)$ and $d(k)$ as state variables, by the following ordinary algebraic equation [1, 2, 6]:

$$d(k) = \max(a(k), d(k-1)) + \tau_k.$$

By replacing the usual operation symbols by those of max-algebra, one can rewrite this equation in its equivalent form as

$$d(k) = \tau_k \otimes a(k) \oplus \tau_k \otimes d(k-1).$$

Let us now suppose that there is a system of n single-server queues with infinite buffers. Furthermore, let $a_i(k)$ and $d_i(k)$ be the k^{th} arrival and departure epochs, and τ_{ik} be the service time of the k^{th} customer for queue i. With the vector-matrix notations

$$\boldsymbol{a}(k) = \begin{pmatrix} a_1(k) \\ \vdots \\ a_n(k) \end{pmatrix}, \qquad \boldsymbol{d}(k) = \begin{pmatrix} d_1(k) \\ \vdots \\ d_n(k) \end{pmatrix}, \qquad \mathcal{T}_k = \begin{pmatrix} \tau_{1k} & & \varepsilon \\ & \ddots & \\ \varepsilon & & \tau_{nk} \end{pmatrix},$$

and the conditions $\boldsymbol{d}(0) = (e, \ldots, e)^T$, and $\boldsymbol{d}(k) = (\varepsilon, \ldots, \varepsilon)^T$ for all $k < 0$, we may represent the dynamics of the whole system by the vector equation

$$\boldsymbol{d}(k) = \mathcal{T}_k \otimes \boldsymbol{a}(k) \oplus \mathcal{T}_k \otimes \boldsymbol{d}(k-1). \tag{3}$$

Note that equation (3) is quite general; it may be appropriate for a variety of single-server queueing systems with infinite buffers, and not just for tandem systems. We will show how this equation can be refined in the case of two particular tandem systems in the next subsections.

3.1 Closed Tandem Systems

Consider a closed system of n single-server queues in tandem, with infinite buffers. In the system, the customers have to pass through the queues consecutively so as to receive service at each server. After their service completion at the n^{th} server, the customers return to the first queue for a new cycle of service. It is assumed that the transition of customers from a queue to the next one requires no time.

There are a finite number of customers circulating through the system. We suppose that at the initial time, all servers are free, whereas the buffer at the i^{th} server contains c_i customers, $i = 1, \ldots, n$. It is easy to see that in this case, the arrival times of customers can be defined as

$$a_i(k) = \begin{cases} d_n(k - c_1), & \text{if } i = 1 \\ d_{i-1}(k - c_i), & \text{if } i = 2, \ldots, n, \end{cases} \tag{4}$$

for all $k = 1, 2, \ldots$.

Let us now assume for simplicity that $c_i = 1$ for all $i = 1, \ldots, n$. With this assumption, one can rewrite (4) in the vector form

$$\boldsymbol{a}(k) = F \otimes \boldsymbol{d}(k-1), \qquad \text{where} \qquad F = \begin{pmatrix} \varepsilon & \cdots & \varepsilon & e \\ e & \ddots & \varepsilon & \varepsilon \\ & \ddots & \ddots & \vdots \\ \varepsilon & & e & \varepsilon \end{pmatrix}.$$

Substitution of this expression for $\boldsymbol{a}(k)$ into (3) leads to

$$\boldsymbol{d}(k) = \mathcal{T}_k \otimes F \otimes \boldsymbol{d}(k-1) \oplus \mathcal{T}_k \otimes \boldsymbol{d}(k-1) = \mathcal{T}_k \otimes (F \oplus E) \otimes \boldsymbol{d}(k-1).$$

We may now describe the dynamics of the closed system under consideration by the linear state equation

$$\boldsymbol{d}(k) = T_k \otimes \boldsymbol{d}(k-1), \tag{5}$$

with the state transition matrix

$$T_k = \mathcal{T}_k \otimes (F \oplus E) = \begin{pmatrix} \tau_{1k} & \varepsilon & \cdots & \varepsilon & \tau_{1k} \\ \tau_{2k} & \tau_{2k} & & \varepsilon & \varepsilon \\ & \ddots & \ddots & & \vdots \\ \varepsilon & \varepsilon & \ddots & \tau_{n-1k} & \varepsilon \\ \varepsilon & \varepsilon & & \tau_{nk} & \tau_{nk} \end{pmatrix}. \tag{6}$$

One can see that equation (5) offers a very convenient way of calculating successive state vectors $\boldsymbol{d}(k)$ in the above system, by performing simple algebraic operations. Moreover, it is not difficult to understand that (5) can be readily extended to represent closed tandem systems with arbitrary c_i, $0 \leq c_i < \infty$. A usual technique based on the employment of a modified state vector $\widetilde{\boldsymbol{d}}(k)$ which integrates several consecutive state vectors of the original model, can be exploited to get (5) as a representation of such systems.

To illustrate, let us suppose that at the initial time, there are $c_i = 2$ customers in the buffer at each queue i, $i = 1, \ldots, n$. In this case, starting from (4), we may define the vector of the k^{th} arrival epochs as

$$\boldsymbol{a}(k) = F \otimes \boldsymbol{d}(k-2),$$

and then rewrite (3) in the form

$$\boldsymbol{d}(k) = \mathcal{T}_k \otimes \boldsymbol{d}(k-1) \oplus \mathcal{T}_k \otimes F \otimes \boldsymbol{d}(k-2).$$

Finally, with the new state vector $\widetilde{\boldsymbol{d}}(k) = \begin{pmatrix} \boldsymbol{d}(k) \\ \boldsymbol{d}(k-1) \end{pmatrix}$, it is not difficult to arrive at the state equation

$$\widetilde{\boldsymbol{d}}(k) = \widetilde{T}_k \otimes \widetilde{\boldsymbol{d}}(k-1),$$

with the state transition matrix

$$\widetilde{T}_k = \begin{pmatrix} \mathcal{T}_k & \mathcal{T}_k \otimes F \\ E & \mathcal{E} \end{pmatrix}.$$

3.2 Open Tandem Systems

The purpose of this subsection is to extend equation (5) to the representation of *open* tandem queueing system models. In a series of single-server queues with infinite buffers, let us assign the first queue for representing an external arrival stream of customers. Each customer that arrives into the system has to pass through queues 2 to n, and then leaves the system.

As in the closed system above, we suppose that τ_{ik} represents the k^{th} service time at queue i, $i = 2, \ldots, n$, whereas τ_{1k} now denotes the inter-arrival time between the k^{th} customer and his predecessor in the external arrival stream. At the initial time, all servers are assumed to be free of customers, and, except for the first server, their buffers are empty; that is, $c_i = 0$, $i = 2, \ldots, n$. Finally, we put $c_1 = \infty$ to provide the model with the infinite arrival stream.

It is clear that we have to define the arrival epochs in the system as

$$a_i(k) = \begin{cases} \varepsilon, & \text{if } i = 1 \\ d_{i-1}(k), & \text{if } i = 2, \ldots, n, \end{cases} \tag{7}$$

for all $k = 1, 2, \ldots$. Proceeding to vector notation, we get

$$\boldsymbol{a}(k) = G \otimes \boldsymbol{d}(k), \qquad \text{where} \qquad G = \begin{pmatrix} \varepsilon & \cdots & \cdots & \varepsilon \\ e & \ddots & & \vdots \\ & \ddots & \ddots & \vdots \\ \varepsilon & & e & \varepsilon \end{pmatrix}.$$

With this vector representation, equation (3) takes the form

$$\boldsymbol{d}(k) = \mathcal{T}_k \otimes G \otimes \boldsymbol{d}(k) \oplus \mathcal{T}_k \otimes \boldsymbol{d}(k-1).$$

The above equation can be considered as an implicit equation in $\boldsymbol{d}(k)$, having the form of (1) with $A = \mathcal{T}_k \otimes G$ and $\boldsymbol{b} = \mathcal{T}_k \otimes \boldsymbol{d}(k-1)$. Moreover, one can readily see that the matrix

$$\mathcal{T}_k \otimes G = \begin{pmatrix} \varepsilon & \cdots & \cdots & \varepsilon \\ \tau_{2k} & \ddots & & \vdots \\ & \ddots & \ddots & \vdots \\ \varepsilon & & \tau_{nk} & \varepsilon \end{pmatrix}$$

looks just like that presented in Subsection 2.2 as an example; therefore, it satisfies the condition of Lemma 1.

By applying Lemma 1, we obtain the solution

$$\boldsymbol{d}(k) = \sum_{i=0}^{n-1}{}_{\oplus} (\mathcal{T}_k \otimes G)^i \otimes \mathcal{T}_k \otimes \boldsymbol{d}(k-1).$$

Clearly, we have arrived at equation (5), where the transition matrix is defined as

$$T_k = \sum_{i=0}^{n-1}{}_{\oplus} (\mathcal{T}_k \otimes G)^i \otimes \mathcal{T}_k.$$

It is not difficult to verify that [11]

$$T_k = \begin{pmatrix} \tau_{1k} & \varepsilon & \varepsilon & \dots & \varepsilon \\ \tau_{2k} \otimes \tau_{1k} & \tau_{2k} & \varepsilon & \dots & \varepsilon \\ \vdots & \vdots & & \ddots & \vdots \\ \tau_{n-1k} \otimes \cdots \otimes \tau_{1k} & \tau_{n-1k} \otimes \cdots \otimes \tau_{2k} & \tau_{n-1k} \otimes \cdots \otimes \tau_{3k} & & \varepsilon \\ \tau_{nk} \otimes \cdots \otimes \tau_{1k} & \tau_{nk} \otimes \cdots \otimes \tau_{2k} & \tau_{nk} \otimes \cdots \otimes \tau_{3k} & \dots & \tau_{nk} \end{pmatrix}. \tag{8}$$

4 Tandem Queues with Finite Buffers and Blocking

Suppose now that the buffers of the servers in the open tandem system described above have limited capacity. Consequently, servers may be blocked according to one of the blocking rules. In this paper, we restrict our consideration to *manufacturing* blocking and *communication* blocking, which are most commonly encountered in practice [1, 2].

Consider an open system with n servers in tandem, and assume the buffer at the i^{th} server, $i = 2, \dots, n$, to have capacity b_i, $0 \leq b_i < \infty$. We suppose that the buffer of the first server, which is the input buffer of the system, is infinite. Below is shown how the dynamics of tandem systems which operate according to both manufacturing and communication blocking rules, can be described by state equation (5).

4.1 Manufacturing Blocking

Let us first suppose that the dynamics of the system follows the manufacturing blocking rule. Under this type of blocking, if upon completion of a service, the i^{th} server sees the buffer of the $(i+1)^{\text{st}}$ server full, the former server cannot be freed and has to remain busy until the $(i+1)^{\text{st}}$ server completes its current service to provide a free space in its buffer. Clearly, since the customers leave the system upon their service completion at the n^{th} server, this server cannot be blocked.

It is not difficult to understand that the dynamics can be described by the ordinary scalar equations [1, 2, 6]

$$\begin{aligned} d_i(k) &= \max(\max(a_i(k), d_i(k-1)) + \tau_{ik}, d_{i+1}(k - b_{i+1} - 1)), \\ &\qquad i = 1, \ldots, n-1, \\ d_n(k) &= \max(a_n(k), d_n(k-1)) + \tau_{nk}, \end{aligned}$$

where $a_i(k),\ i = 1, \ldots, n$, are still defined by (7). With max-algebra, one can readily rewrite these equations as [11]

$$\begin{aligned} d_i(k) &= \tau_{ik} \otimes a_i(k) \oplus \tau_{ik} \otimes d_i(k-1) \oplus d_{i+1}(k - b_{i+1} - 1), \\ &\qquad i = 1, \ldots, n-1, \\ d_n(k) &= \tau_{nk} \otimes a_n(k) \oplus \tau_{nk} \otimes d_n(k-1). \end{aligned}$$

Assuming $b_i = 0,\ i = 2, \ldots, n$, for simplicity, we get the above set of equations in the vector form

$$\begin{aligned} \boldsymbol{d}(k) &= \mathcal{T}_k \otimes \boldsymbol{a}(k) \oplus \mathcal{T}_k \otimes \boldsymbol{d}(k-1) \oplus G^T \otimes \boldsymbol{d}(k-1) \\ &= \mathcal{T}_k \otimes \boldsymbol{a}(k) \oplus (\mathcal{T}_k \oplus G^T) \otimes \boldsymbol{d}(k-1), \end{aligned}$$

where G^T denotes the transpose of the above introduced matrix G. With $\boldsymbol{a}(k) = G \otimes \boldsymbol{d}(k)$, we have the equation

$$\boldsymbol{d}(k) = \mathcal{T}_k \otimes G \otimes \boldsymbol{d}(k) \oplus (\mathcal{T}_k \oplus G^T) \otimes \boldsymbol{d}(k-1).$$

As in the case of the open tandem system with infinite buffers, we may apply Lemma 1 to solve this equation for $\boldsymbol{d}(k)$, and obtain

$$\boldsymbol{d}(k) = \sum_{i=0}^{n-1}{}_{\oplus} (\mathcal{T}_k \otimes G)^i \otimes (G^T \oplus \mathcal{T}_k) \otimes \boldsymbol{d}(k-1),$$

which equals (5) with the transition matrix

$$T_k = \sum_{i=0}^{n-1}{}_{\oplus} (\mathcal{T}_k \otimes G)^i \otimes (G^T \oplus \mathcal{T}_k).$$

Calculation of T_k [11] leads us to

$$T_k = \begin{pmatrix} \tau_{1k} & e & \varepsilon & \ldots & \varepsilon \\ \tau_{2k} \otimes \tau_{1k} & \tau_{2k} & e & & \varepsilon \\ \vdots & \vdots & & \ddots & \\ \tau_{n-1k} \otimes \cdots \otimes \tau_{1k} & \tau_{n-1k} \otimes \cdots \otimes \tau_{2k} & \tau_{n-1k} \otimes \cdots \otimes \tau_{3k} & & e \\ \tau_{nk} \otimes \cdots \otimes \tau_{1k} & \tau_{nk} \otimes \cdots \otimes \tau_{2k} & \tau_{nk} \otimes \cdots \otimes \tau_{3k} & \ldots & \tau_{nk} \end{pmatrix}. \tag{9}$$

Note that the matrices in (8) and (9) differ only in the elements of the upper diagonal adjacent to the main diagonal, which become equal to e in (9).

Let us now derive equation (5) for the system with the capacity of each buffer $b_i = 1,\ i = 2, \ldots, n$. Clearly, the dynamics of the system can be described by the equation

$$\boldsymbol{d}(k) = \mathcal{T}_k \otimes G \otimes \boldsymbol{d}(k) \oplus \mathcal{T}_k \otimes \boldsymbol{d}(k-1) \oplus G^T \otimes \boldsymbol{d}(k-2).$$

The application of Lemma 1 leads us to the equation

$$\boldsymbol{d}(k) = \sum_{i=0}^{n-1}{}_{\oplus} (\mathcal{T}_k \otimes G)^i \otimes (\mathcal{T}_k \otimes \boldsymbol{d}(k-1) \oplus G^T \otimes \boldsymbol{d}(k-2)).$$

Finally, with the notations

$$\widetilde{\boldsymbol{d}}(k) = \begin{pmatrix} \boldsymbol{d}(k) \\ \boldsymbol{d}(k-1) \end{pmatrix} \qquad \text{and} \qquad S_k = \sum_{i=0}^{n-1}{}_{\oplus} (\mathcal{T}_k \otimes G)^i,$$

we arrive at (5), taking the form

$$\widetilde{\boldsymbol{d}}(k) = \begin{pmatrix} S_k \otimes \mathcal{T}_k & S_k \otimes G^T \\ E & \mathcal{E} \end{pmatrix} \otimes \widetilde{\boldsymbol{d}}(k-1).$$

4.2 Communication Blocking

We now turn to a brief discussion of systems operating under the communication blocking rule. This type of blocking requires a server not to initiate service of a customer if the buffer of the next server is full. In this case, the server remains unavailable until the current service at the next server is completed.

Let us suppose that the above system follows communication blocking. The dynamics of this system can be described by the equations [1, 6]

$$\begin{aligned} d_i(k) &= \max(a_i(k), d_i(k-1), d_{i+1}(k - b_{i+1} - 1)) + \tau_{ik}, \\ &\qquad i = 1, \ldots, n-1, \\ d_n(k) &= \max(a_n(k), d_n(k-1)) + \tau_{nk}, \end{aligned}$$

or, equivalently, by the max-algebra equations

$$\begin{aligned} d_i(k) &= \tau_{ik} \otimes a_i(k) \oplus \tau_{ik} \otimes d_i(k-1) \oplus \tau_{ik} \otimes d_{i+1}(k - b_{i+1} - 1), \\ &\qquad i = 1, \ldots, n-1, \\ d_n(k) &= \tau_{nk} \otimes a_n(k) \oplus \tau_{nk} \otimes d_n(k-1). \end{aligned}$$

For a particular system with $b_i = 0,\ i = 2, \ldots, n$, we can write in the same manner as for manufacturing blocking

$$\begin{aligned} \boldsymbol{d}(k) &= \mathcal{T}_k \otimes \boldsymbol{a}(k) \oplus \mathcal{T}_k \otimes \boldsymbol{d}(k-1) \oplus \mathcal{T}_k \otimes G^T \otimes \boldsymbol{d}(k-1) \\ &= \mathcal{T}_k \otimes G \otimes \boldsymbol{d}(k) \oplus \mathcal{T}_k \otimes (E \oplus G^T) \otimes \boldsymbol{d}(k-1). \end{aligned}$$

Furthermore, we have the solution

$$\boldsymbol{d}(k) = \sum_{i=0}^{n-1}{}_{\oplus} (\mathcal{T}_k \otimes G)^i \otimes \mathcal{T}_k \otimes (E \oplus G^T) \otimes \boldsymbol{d}(k-1),$$

which takes the form of (5) with the matrix

$$\begin{aligned} T_k &= \sum_{i=0}^{n-1}{}_{\oplus} (\mathcal{T}_k \otimes G)^i \otimes \mathcal{T}_k \otimes (E \oplus G^T) \\ &= \begin{pmatrix} \tau_{1k} & \tau_{1k} & \varepsilon & \dots & \varepsilon \\ \tau_{2k} \otimes \tau_{1k} & \tau_{2k} \otimes \tau_{1k} & \tau_{2k} & & \varepsilon \\ \vdots & \vdots & \vdots & \ddots & \\ \tau_{n-1k} \otimes \cdots \otimes \tau_{1k} & \tau_{n-1k} \otimes \cdots \otimes \tau_{1k} & \tau_{n-1k} \otimes \cdots \otimes \tau_{2k} & \dots & \tau_{n-1k} \\ \tau_{nk} \otimes \cdots \otimes \tau_{1k} & \tau_{nk} \otimes \cdots \otimes \tau_{1k} & \tau_{nk} \otimes \cdots \otimes \tau_{2k} & \dots & \tau_{nk} \otimes \tau_{n-1k} \end{pmatrix}. \end{aligned} \tag{10}$$

Let us finally consider the system with the capacity of each buffer $b_i = 1$, $i = 2, \dots, n$. As for manufacturing blocking, we may represent the dynamics of the system by the equation

$$\widetilde{\boldsymbol{d}}(k) = \begin{pmatrix} S_k \otimes \mathcal{T}_k & S_k \otimes \mathcal{T}_k \otimes G^T \\ E & \mathcal{E} \end{pmatrix} \otimes \widetilde{\boldsymbol{d}}(k-1).$$

5 Representation of Tandem System Performance

In this section, we show how performance measures of open tandem systems may be represented based on the max-algebra models. We consider the measures representing the system time and the waiting time of customers, and give the corresponding linear max-algebra equations which allow us to describe and calculate these criteria in a simple way. The ordinary representation of other measures which one normally chooses in the analysis of queueing systems, including the utilization of a server, the number of customers at a queue, and the queue length, involves the operation of division [1, 3, 4, 6], and they therefore cannot be expressed through linear equations in max-algebra.

5.1 The System Time of Customers

For the open tandem system with infinite buffers described above, let us define the vector of the system (sojourn) times of the k^{th} customer as $\boldsymbol{s}(k) = (s_1(k), \dots, s_n(k))^T$, where $s_i(k)$ denotes the time required for the customer to pass through all queues up to and including queue i. Since the first queue represents the external arrival stream of customers, we do not have

to include the time spent in this queue, and we therefore have $s_1(k) = 0$. Furthermore, the above definition of the system time leads to the equations

$$s_i(k) = d_i(k) - d_1(k), \quad i = 1, \ldots, n.$$

With max-algebra vector notations, we may rewrite these equations as

$$\boldsymbol{s}(k) = \boldsymbol{d}(k) \otimes d_1^{-1}(k). \tag{11}$$

It follows from (5) and the equality $d_1(k) = d_1(k-1) \otimes \tau_{1k}$ that

$$\boldsymbol{s}(k) = T_k \otimes \boldsymbol{d}(k-1) \otimes d_1^{-1}(k-1) \otimes \tau_{1k}^{-1}.$$

By applying (11) with k replaced by $k-1$, we finally have

$$\boldsymbol{s}(k) = U_k \otimes \boldsymbol{s}(k-1), \tag{12}$$

where $U_k = \tau_{1k}^{-1} \otimes T_k$.

As it is easy to see, the above dynamic equation is appropriate for representing the system time in open tandem systems with both infinite and finite buffers. For particular systems, equation (12) will differ only in the matrix T_k, and thus in the matrix U_k, inherent in their associated dynamic state equations. Finally, note that a lower triangular transition matrix T_k will result in a matrix U_k of the same kind. One can consider the open tandem system with infinite buffers as an example.

5.2 The Waiting Time of Customers

Consider the open tandem system with infinite buffers once again. The system time of a customer in this system consists of his service time and the waiting time. Therefore, introducing the symbol $w_i(k)$ for the k^{th} customer to denote the total time spent on waiting for service at the queues from 1 to i, we have the equations

$$\begin{aligned} s_1(k) &= w_1(k) = 0, \\ s_i(k) &= w_i(k) + \sum_{j=2}^{i} \tau_{jk}, \quad i = 2, 3, \ldots, n. \end{aligned}$$

In order to represent the relation between the system and waiting times in max-algebra vector form, let us further introduce the vector $\boldsymbol{w}(k) = (w_1(k), \ldots, w_n(k))^T$, and the diagonal matrix

$$P_k = \begin{pmatrix} \tau_{1k} & \varepsilon & \cdots & \varepsilon \\ \varepsilon & \tau_{1k} \otimes \tau_{2k} & & \varepsilon \\ \vdots & & \ddots & \\ \varepsilon & \varepsilon & & \tau_{1k} \otimes \cdots \otimes \tau_{nk} \end{pmatrix}.$$

Clearly, we may now write

$$\boldsymbol{s}(k) = \tau_{1k}^{-1} \otimes P_k \otimes \boldsymbol{w}(k).$$

Since the multiplicative inverse exists for the matrix P_k, it is not difficult to resolve the above equation for $\boldsymbol{w}(k)$:

$$\boldsymbol{w}(k) = \tau_{1k} \otimes P_k^{-1} \otimes \boldsymbol{s}(k).$$

With (12), we successively obtain

$$\begin{aligned}\boldsymbol{w}(k) &= \tau_{1k} \otimes P_k^{-1} \otimes \tau_{1k}^{-1} \otimes T_k \otimes \boldsymbol{s}(k-1) \\ &= P_k^{-1} \otimes T_k \otimes \tau_{1k-1}^{-1} \otimes P_{k-1} \otimes \big(\tau_{1k-1} \otimes P_{k-1}^{-1} \otimes \boldsymbol{s}(k-1)\big) \\ &= \tau_{1k-1}^{-1} \otimes P_k^{-1} \otimes T_k \otimes P_{k-1} \otimes \boldsymbol{w}(k-1).\end{aligned}$$

Finally, we arrive at the dynamic equation

$$\boldsymbol{w}(k) = V_{k,k-1} \otimes \boldsymbol{w}(k-1), \tag{13}$$

with the transition matrix $V_{k,k-1} = \tau_{1k-1}^{-1} \otimes P_k^{-1} \otimes T_k \otimes P_{k-1}$.

Note that, in general, equation (13) can be extended to tandem queues with finite buffers and blocking. In that case, however, the quantities $w_i(k)$ will include not only the time spent on waiting for service, but also the blocking time of servers.

6 Serial and Parallel Simulation of Tandem Queues

In this section, we briefly discuss serial and parallel simulation algorithms based on the algebraic models presented above, and outline their performance. We take, as the starting point, the state equation

$$\boldsymbol{d}(k) = T_k \otimes \boldsymbol{d}(k-1). \tag{14}$$

with the matrix T_k defined by (8). Clearly, calculations with other matrices, including performance evaluation via equations (12) and (13), will normally differ little in complexity.

Furthermore, we assume as in [1, 2, 7] that the service times τ_{ik}, which are normally defined in queueing system simulation as realizations of given random variables, are available for all $i = 1, \ldots, n$ and $k = 1, 2 \ldots$, when required. We will therefore concentrate only on procedures of evaluating the system state vectors, based on (14).

6.1 Simulation with a Scalar Processor

Let us first consider time and memory requirements when only one scalar processor is available for simulation of tandem systems. The related serial algorithm consists of consecutive steps, evaluating new state vectors. The k^{th} step involves determination of the transition matrix T_k and multiplication of this matrix by the vector $\boldsymbol{d}(k-1)$ to produce the vector $\boldsymbol{d}(k)$. However, particular algorithms designed for computation with matrices (6), (8), (9), and (10) may execute the step in different ways, according to the structure of the matrices, so as to reduce time and memory costs.

Assume that a system is simulated until the K^{th} service completion at server n, and denote the overall number of the operations $\oplus$ and $\otimes$ to be performed within the k^{th} step to evaluate the matrix T_k and to compute the product $T_k \otimes \boldsymbol{d}(k-1)$ respectively by N_1 and N_2. In that case, the entire simulation algorithm will require $N = K(N_1 + N_2)$ operations, ignoring index manipulations. Finally, we denote by M the number of memory locations involved in the computations.

Let us now consider the open tandem system with infinite buffers, and denote the entries of its transition matrix T_k by $t_{ij}^{(k)}$, $i = 1, \ldots, n$, $j = 1, \ldots, n$. Taking into account the lower triangular form of the matrix T_k defined by (8), a serial algorithm for calculating K successive vectors $\boldsymbol{d}(k)$ may be readily designed as follows.

Algorithm 1.

For $i = 1, \ldots, n$, **do** $d_i(0) \longleftarrow \varepsilon$.
For $k = 1, \ldots, K$, **do**
  for $i = 1, \ldots, n$, **do**
    $t_{ii}^{(k)} \longleftarrow \tau_{ik}$;
    for $j = 1, \ldots, i-1$, **do**
      $t_{ij}^{(k)} \longleftarrow t_{i-1j}^{(k)} \otimes \tau_{ik}$;
  for $i = 1, \ldots, n$, **do**
    $d_i(k) \longleftarrow t_{i1}^{(k)} \otimes d_i(k-1)$;
    for $j = 2, \ldots, i$, **do**
      $d_i(k) \longleftarrow d_i(k) \oplus t_{ij}^{(k)} \otimes d_j(k-1)$.

It is not difficult to see that for the algorithm, $N_1 = n(n+1)/2$ and $N_2 = n^2$ operations are required; it entails $M = n(n+5)/2$ memory locations. With the total number of operations $N = O(n(3n+1)K/2)$ involved in calculating K state vectors, the algorithm proves to have a performance comparable to other serial simulation procedures [1, 7]. As an example, the serial algorithm in [7], designed for simulation of open tandem systems with infinite buffers can be considered. This algorithm allows one to compute the K^{th} departure time at queue n in $O(2K(n+1))$ operations; however, it is

actually a scalar algorithm intended to obtain only the value of $d_n(k)$ rather than the whole vector $\boldsymbol{d}(k)$. Therefore, one has to multiply its performance criterion by n so as to provide a proper basis for a comparison between algorithms. It should be noted in conclusion that calculations with matrix (6) can be performed using only $N = O(2Kn)$ operations and $M = O(3n)$ memory locations.

6.2 Simulation with a Vector Processor

Suppose now that simulation is executed on a vector processor equipped with vector registers of a length large enough for the processing of n-vectors. With the notations $\boldsymbol{t}_i^{(k)} = (t_{i1}^{(k)}, \ldots, t_{in}^{(k)})$, $i = 1, \ldots, n$, we may write the following vector modification of Algorithm 1:

Algorithm 2.

$\boldsymbol{d}(0) \longleftarrow (\varepsilon, \ldots, \varepsilon)^T$.
For $k = 1, \ldots, K$, **do**
  $T_k \longleftarrow \mathcal{E}$;
  for $i = 1, \ldots, n$, **do**
    $t_{ii}^{(k)} \longleftarrow \tau_{ik}$;
    $\boldsymbol{t}_i^{(k)} \longleftarrow \boldsymbol{t}_{i-1}^{(k)} \otimes \tau_{ik}$;
    $d_i(k) \longleftarrow \boldsymbol{t}_i^{(k)} \otimes \boldsymbol{d}(k-1)$.

First note that implementation of the vector processor makes it possible to compute matrix (8) by performing only $N_1 = n$ vector operations. Furthermore, evaluation of each element of the vector $\boldsymbol{d}(k)$ from (14) actually involves both componentwise addition of a row of the matrix T_k to the vector $\boldsymbol{d}(k-1)$, and determination of the maximum over the elements of this vector sum.

With the vector processor, one can add two vectors together in a single operation. It follows from the triangular form of (8) that to compute the i^{th} element of $\boldsymbol{d}(k)$, one actually has to perform no more than i maximizations. By applying the *recursive doubling method* [13], we may obtain the maximum over i consecutive elements of a vector in $\log_2 i$ operations. For all entries of the vector $\boldsymbol{d}(k)$, we therefore get $N_2 = n + \log_2 1 + \cdots + \log_2 n = n + \log_2(n!)$.

Finally, evaluation of K state vectors requires $N = O(K(\log_2(n!) + 2n))$ operations of a vector processor. As is easy to see, vector computations allow us to achieve the speedup $S_v = O(n(3n+1)/(\log_2(n!)/2 + n))$ in relation to the sequential procedure discussed in the previous subsection.

6.3 Parallel Simulation of Tandem Systems

We conclude this section with the discussion of a parallel simulation algorithm intended for implementation on single instruction, multiple data

(SIMD) parallel processors. Other parallel algorithms based on the algebraic models can be found in [2].

Let P be the number of processors available, and $P \geq n$. The algorithm consists of $L = \lceil K/P \rceil$ steps, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x, and it can be described as follows.

Algorithm 3.

$\boldsymbol{d}(0) \longleftarrow (\varepsilon, \ldots, \varepsilon)^T$.
for $l = 1, \ldots, L,$ **do**
  in parallel, for $i = (l-1)P + 1, (l-1)P + 2, \ldots, \min(lP, K),$ **do**
    evaluate T_i;
  for $i = (l-1)P + 1, (l-1)P + 2, \ldots, \min(lP, K),$ **do**
    in parallel, for $j = 1, \ldots, n,$ **do** $d_j(i) \longleftarrow \boldsymbol{t}_j^{(i)} \otimes \boldsymbol{d}(i-1)$.

To estimate the performance of this algorithm, first note that each step starts with parallel evaluation of P consecutive transition matrices, which entails $N_1 = n(n+1)/2$ parallel operations. Next follows the determination of P consecutive state vectors, with each vector evaluated in parallel by computing its elements on separate processors. Since evaluation of a vector element involves n ordinary additions and the same number of maximizations, one has to perform $2n$ parallel operations so as to determine the whole vector. Therefore, we have $N_2 = 2Pn$ operations required for computing P vectors.

Finally, the entire algorithm entails $N = L(n(n+1)/2 + 2Pn)$ parallel operations. It is not difficult to understand that with $P \geq n$, we get $N = O(3Kn/2)$ as $K \to \infty$. Moreover, by comparison with the performance of the above sequential procedure, one can conclude that for $P = n$ and K sufficiently large, the parallel algorithm achieves the speedup $S_P = O(3P/5)$.

7 Conclusions

The max-algebra approach provides a very convenient way of representing tandem queueing system models. The algebraic models describe the dynamics of the systems through linear max-algebra equations similar to those being studied in the conventional linear systems theory. The approach therefore offers the potential for extending classical results of both conventional linear algebra and systems theory to analyze queueing systems. Moreover, it provides the basis for applying methods and algorithms of numerical algebra to the development of efficient procedures for the queueing systems simulation, including the algorithms designed for implementation on parallel and vector processors.

Since the closed form representation of the dynamics of more complicated queueing systems including the $G/G/m$ queue and queueing networks, normally involves three operations, namely addition, maximization, and minimization (see, e.g., [5, 6]), these systems cannot be described in terms of max-algebra. In that case, however, appropriate algebraic representations can be obtained using *minimax algebra* [8, 10].

References

[1] L. Chen and C.-L. Chen,
"A fast simulation approach for tandem queueing systems," in *1990 Winter Simulation Conference Proceedings*, O. Balci, R. P. Sadowski, and R. E. Nance, eds., pp. 539–546. IEEE, 1990.

[2] A. G. Greenberg, B. D. Lubachevsky, and I. Mitrani, "Algorithms for unboundedly parallel simulations,"
ACM Trans. Comput. Syst. **9** no. 3, (1991) 201–221.

[3] N. Krivulin, "Unbiased estimates for gradients of stochastic network performance measures," *Acta Appl. Math.* **33** (1993) 21–43, `arXiv:1210.5418 [math.OC]`.

[4] N. K. Krivulin, "Optimization of complex systems for simulation modeling," *Vestnik Leningrad Univ. Math.* **23** no. 2, (1990) 64–67.

[5] N. K. Krivulin, "A recursive equations based representation for the g/g/m queue," *Appl. Math. Lett.* **7** no. 3, (1994) 73–77, `arXiv:1210.6012 [math.OC]`.

[6] N. K. Krivulin, "Recursive equations based models of queueing systems," in *Proc. Europ. Simulation Symp. ESS 94*, pp. 252–256. SCSI, San Diego, CA, 1994. `arXiv:1210.7445 [math.NA]`.

[7] S. M. Ermakov and N. K. Krivulin, "Efficient algorithms for tandem queueing system simulation," *Appl. Math. Lett.* **7** no. 6, (1994) 45–49, `arXiv:1210.6378 [math.NA]`.

[8] R. Cuninghame-Green, *Minimax Algebra*, vol. 166 of *Lecture Notes in Economics and Mathematical Systems*. Springer, Berlin, 1979.

[9] G. Cohen, P. Moller, J.-P. Quadrat, and M. Viot, "Algebraic tools for the performance evaluation of discrete event systems,"
Proc. IEEE **77** no. 1, (January, 1989) 39–85.

[10] R. A. Cuninghame-Green, "Minimax algebra and applications,"
Fuzzy Sets and Syst. **41** no. 3, (1991) 251–267.

[11] N. K. Krivulin, "Using max-algebra linear models in the representation of queueing systems," in *Proc. 5th SIAM Conf. on Applied Linear Algebra*, J. G. Lewis, ed., pp. 155–160. SIAM, Philadelphia, 1994. `arXiv:1210.6019 [math.OC]`.

[12] G. J. Olsder, "About difference equations, algebras and discrete events," vol. 31 of *CWI syllabus*, pp. 180–209. CWI, Amsterdam, 1992.

[13] J. M. Ortega, *Introduction to Parallel and Vector Solution of Linear Systems*. Frontiers in Computer Science. Plenum Press, New York, 1988.